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On Edge-Gracefulness of Wheel Graphs

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Abstract

In this paper, we present a complete proof that W_3 is the only edge-graceful wheel graph; a result that first appeared in the published work of S.Venkatesan and P.Sekar in 2017. Particularly, we first discuss the results of S.Venkatesan and P.Sekar, regarding the edge-gracefulness of wheel graphs, highlighting a point for improvement to their proof that W_3 is the only edge-graceful wheel graph. We then provide a complete proof of their result using the concepts of divisibility and Diophantine equations. We end the paper by giving some future works, related to edge-graceful labeling.

Keywords: Graceful Labeling, Edge-graceful Labeling, Wheel Graph, Diophantine Equation.

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1. Introduction

The concept of graph labeling was first introduced in the mid-1960's. A graph labeling is an assignment of integers to the vertices or edges, or both, of a graph under certain conditions [1]. Within the last 60 years, over 200 types or variations of graph labeling have been studied and about 2500 articles have been published [2]. A graceful labeling or graceful numbering is a type of labeling of a graph on m edges in which the vertices are labeled with a subset of distinct non-negative integers from 0 to m and each edge of the graph is uniquely labeled by the absolute difference between the labels of the vertices incident to it. If the resulting labeling of graph edges run from 1 to m inclusive, it is a graceful labeling and the graph is said to be a graceful graph [3].On the other hand, the concept of edge-gracefulness and edge-graceful graphs was defined and introduced by Sheng-Ping in [4]. Edge-graceful labeling is considered as a reversal of graceful labeling, because it labels the edges first, then the labels of the vertices would depend on the labels of the edges incident to them. That is, a graph G with p vertices and q edges, is said to be edge-graceful if the edges can be labeled from 1 through q, in such a way that the labels induced on the vertices by summing over incident edges modulo p are distinct. In 2017, S.Venkatesan and P.Sekar wrote a paper regarding the edge-gracefulness of wheel graphs. They used the Lo's Theorem and Microsoft Excel to prove the main result of their paper, which claims that W_3 is the only edge-graceful wheel graph [3]. Motivated by their work, this research which aims to provide a complete proof of their result was created.

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2. Preliminaries

2.1. Preliminary Graph Theory Concepts

Throughout this paper a graph with vertex set V and edge set E will be denoted by $G(V, E)$. Moreover, we denote a graph with p number of vertices and q number of edges as (p, q)-graph. For readers who are unfamiliar with graph theory concepts and notations that were briefly stated in this paper, we refer them to some standard graph theory references such as Gallian's "A Dynamic Survey of Graph Labeling" [5] and Gross' "Graph theory and its applications" [6]. The following essential terms and results from Wolfram MathWorld [7] (aside from the concept of edge-graceful labeling) will be encountered and used in the succeeding sections of this article.

Definition 2.1 (Edge Graceful Graph). A (p, q)-graph $G(V, E)$ is called edge-graceful if there exists a bijection $f : E \to \{1, 2, ..., q\}$, such that the induced mapping $f^+ : V \to \mathbb{Z}_p$, defined by

$$f^+(u) \equiv \Big(\sum_{v \in N(u)} f(uv) \Big) (\text{mod } p)$$

is also a bijection.

We note that $\mathbb{Z}_p$ is the set of integers $\{0, 1, 2, .., (p - 1)\}$, $N(u)$ refers to the neighborhood of the vertex u, and $f(uv)$ is the label of the edge with endpoints u and v in G.

Figure 1 below, shows an example of an edge-graceful labeling of a graph. First, the graph involved, has 5 edges and 5 vertices. By following Definition 2.1, the edges of the graph was labeled using distinct integers from 1 to 5, and the induced labels of the vertices are distinct integers from 0 to 4.

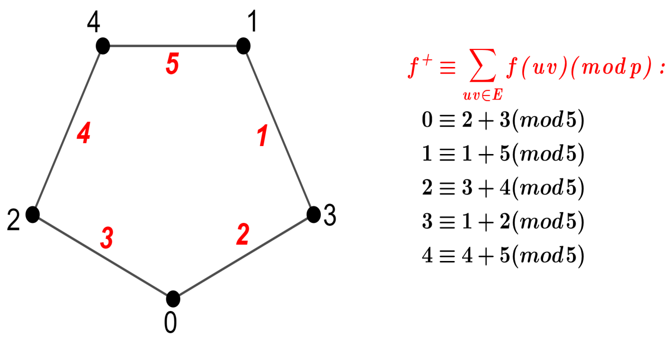

Figure 1: An edge-graceful labeling of cycle graph C_5.

Definition 2.2. A wheel graph is the $(n + 1, 2n)$-graph which is formed from adding the middle vertex and edges connecting the new vertex to every vertex of a cycle C_n. Wheels are denoted by W_n, where n refers to the number of vertices in the outer cycle.

Some examples of wheel graphs are shown in Figure 2.

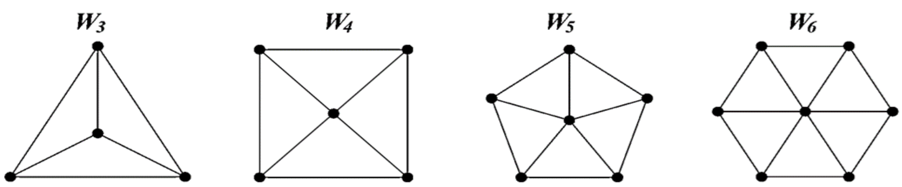

Figure 2: Examples of usual fan graphs.

2.2. Essential Results

This subsection discusses the important concepts and results that will be utilized in this paper such as the Lo's theorem and the general quadratic Diophantine equation. Since most of the concepts here are mostly number-theoretic, we refer readers who are unfamiliar with number theory to Burton's book "Elementary Number Theory" [8].

Theorem 2.3 (Lo's Theorem[9]). If a graph G with p vertices and q edges is edge-graceful, then

$$p\Big|\left(q^2+q-\frac{p(p-1)}{2}\right).$$

Definition 2.4 (General Quadratic Diophantine Equation). Diophantine equations are equations, where only integer solutions are allowed. A general quadratic Diophantine equation is an equation of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$, where x and y are both integers.

Lemma 2.5 (Tamang [10]). The quadratic Diophantine equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{2.1}$$

where a, b, c, d, e, and f are integer coefficients, and x and y are the unknown variables can be reduced to

$$X^2 - DY^2 = N, \tag{2.2}$$

where $X = Dy + E$, $D = b^2 - 4ac$, $E = bd - 2ae$, $Y = 2ax + by + d$, $N = E^2 - DF$, and $F = d^2 - 4af$.

Remark 2.6. Observe that if we have solution X, Y in equation (2.2), then there are integers x and y, such that

$$X = Dy + E \quad \rightarrow \quad y = \frac{X-E}{D}$$

and

$$Y = 2ax + by + d \quad \rightarrow \quad x = \frac{Y-by-d}{2a}$$

where x and y are the integer solutions for equation (2.1).

Remark 2.7. All the Diophantine equations involved in this study has $c = 0$. Thus the equation (2.2) will just be written as

$$X^2 - (bY)^2 = N. \tag{2.3}$$

Observe that we can factor the left side of the equation (2.3) as $(X + bY)(X - bY)$. Moreover, if the pair N_1 and N_2 is a factor pair of N, then we will have linear system of equation,

$$(X + bY) = N_1$$
$$(X - bY) = N_2$$

Now, solving for X and Y, we will have,

$$X = \frac{N_1+N_2}{2} \quad \text{and} \quad Y = \frac{N_1-N_2}{2b}.$$

3. Results

3.1. Exposition of Venkatesan and Sekar's Result

We now present the result of S. Venkatesan and P. Sekar.

Theorem 3.1 (Venkatesan and Sekar, 2017). *Among all wheel graphs W_n, only W_3 is edge-graceful.*

Proof. (by Venkatesan and Sekar) First, recall that a wheel graph W_n has $n+1$ vertices and $2n$ edges. If we substitute $p = n+1$ and $q = 2n$ in Theorem 2.3, we arrived at

$$(n+1)\Big|\left((2n)^2 + 2n - \frac{(n+1)(n)}{2}\right).$$

The divisibility condition above can be further simplified into

$$\left(\frac{7n^2+3n}{2(n+1)}\right), \text{is an integer.}$$

Using Microsoft Excel-2021, Venkatesan and Sekar determined which values of n from 1 to 100 will satisfy $\left(\frac{7n^2+3n}{2(n+1)}\right) \in \mathbb{Z}$. We further extended it and consider the values of n from 1 to 200, as shown in Figure 3.

Figure 3 shows that from 1 to 200, only $n = 3$ will make $\left(\frac{7n^2+3n}{2(n+1)}\right)$ an integer. At this point, they came up with the conclusion that W_3 is the only edge-graceful wheel graph. □

Suggested Improvements

Since the method only considered limited values of n, a method that can consider all positive integer n must be used. Furthermore, an actual example of edge-graceful labeling of W_3 must be provided to complete the proof. This is because, based on the statement of Lo's theorem, the existence of a positive integers p and q such that $p\Big|\left(q^2 + q - \frac{p(p-1)}{2}\right)$ do not guarantee us that the (p, q)-graph is edge-graceful. Rather, the contrapositive of the statement provides us a sieving method as to what (p, q)-graphs are not edge-graceful.

3.2. Proof that W_3 is the only Edge-graceful Wheel Graph

We now complete the proof that W_3 is the only Edge-graceful Wheel Graph.

Proof. From the initial steps of Venkatesan and Sekar, they showed that if the wheel graph W_n is edge-graceful, then

$$\left(\frac{7n^2+3n}{2(n+1)}\right) \in \mathbb{Z}.$$

This means,

$$\begin{aligned} 7n^2 + 3n &= 2(n+1)k \\ &= 2nk + 2k, \quad \text{for some } k \in \mathbb{Z}. \end{aligned}$$

Thus, we solve for all the possible integer values of n and k in the Diophantine equation,

$$7n^2 + 3n - 2nk - 2k = 0 \tag{3.1}$$

where $n, k \in \mathbb{Z}$. This Diophantine equation is of the form,

$$an^2 + bnk + ck^2 + dn + ek + f = 0,$$

where $a = 7$, $b = -2$, $c = 0$, $d = 3$, $e = -2$, and $f = 0$.

n	(7n²+3n)/(2(n+1))	n	(7n²+3n)/(2(n+1))	n	(7n²+3n)/(2(n+1))	n	(7n²+3n)/(2(n+1))	n	(7n²+3n)/(2(n+1))
1	2.5	41	141.547619	81	281.52439	121	421.516393	161	561.512346
2	5.66666667	42	145.046512	82	285.024096	122	425.01626	162	565.01227
3	9	43	148.545455	83	288.52381	123	428.516129	163	568.512195
4	12.4	44	152.044444	84	292.023529	124	432.016	164	572.012121
5	15.8333333	45	155.543478	85	295.523256	125	435.515873	165	575.512048
6	19.2857143	46	159.042553	86	299.022989	126	439.015748	166	579.011976
7	22.75	47	162.541667	87	302.522727	127	442.515625	167	582.511905
8	26.2222222	48	166.040816	88	306.022472	128	446.015504	168	586.011834
9	29.7	49	169.54	89	309.522222	129	449.515385	169	589.511765
10	33.1818182	50	173.039216	90	313.021978	130	453.015267	170	593.011696
11	36.6666667	51	176.538462	91	316.521739	131	456.515152	171	596.511628
12	40.1538462	52	180.037736	92	320.021505	132	460.015038	172	600.011561
13	43.6428571	53	183.537037	93	323.521277	133	463.514925	173	603.511494
14	47.1333333	54	187.036364	94	327.021053	134	467.014815	174	607.011429
15	50.625	55	190.535714	95	330.520833	135	470.514706	175	610.511364
16	54.1176471	56	194.035088	96	334.020619	136	474.014599	176	614.011299
17	57.6111111	57	197.534483	97	337.520408	137	477.514493	177	617.511236
18	61.1052632	58	201.033898	98	341.020202	138	481.014388	178	621.011173
19	64.6	59	204.533333	99	344.52	139	484.514286	179	624.511111
20	68.0952381	60	208.032787	100	348.019802	140	488.014184	180	628.01105
21	71.5909091	61	211.532258	101	351.519608	141	491.514085	181	631.510989
22	75.0869565	62	215.031746	102	355.019417	142	495.013986	182	635.010929
23	78.5833333	63	218.53125	103	358.519231	143	498.513889	183	638.51087
24	82.08	64	222.030769	104	362.019048	144	502.013793	184	642.010811
25	85.5769231	65	225.530303	105	365.518868	145	505.513699	185	645.510753
26	89.0740741	66	229.029851	106	369.018692	146	509.013605	186	649.010695
27	92.5714286	67	232.529412	107	372.518519	147	512.513514	187	652.510638
28	96.0689655	68	236.028986	108	376.018349	148	516.013423	188	656.010582
29	99.5666667	69	239.528571	109	379.518182	149	519.513333	189	659.510526
30	103.064516	70	243.028169	110	383.018018	150	523.013245	190	663.010471
31	106.5625	71	246.527778	111	386.517857	151	526.513158	191	666.510417
32	110.060606	72	250.027397	112	390.017699	152	530.013072	192	670.010363
33	113.558824	73	253.527027	113	393.517544	153	533.512987	193	673.510309
34	117.057143	74	257.026667	114	397.017391	154	537.012903	194	677.010256
35	120.555556	75	260.526316	115	400.517241	155	540.512821	195	680.510204
36	124.054054	76	264.025974	116	404.017094	156	544.012739	196	684.010152
37	127.552632	77	267.525641	117	407.516949	157	547.512658	197	687.510101
38	131.051282	78	271.025316	118	411.016807	158	551.012579	198	691.01005
39	134.55	79	274.525	119	414.516667	159	554.5125	199	694.51
40	138.04878	80	278.024691	120	418.016529	160	558.012422	200	698.00995

Figure 3: Checking which Values of n from 1 to 200 will satisfy $\left(\frac{7n^2+3n}{2(n+1)}\right) \in \mathbb{Z}$, using Microsoft Excel-2021.

Using the transformation stated in Lemma 2.5, the equation (3.1) can be reduced to

$$X^2 - 4Y^2 = 448 \tag{3.2}$$

where $Y = 2an + bk + d = 14n - 2k + 3$ and $X = Dk + E = 4k + 22$.
By factoring the left side of the equation 3.2, we will have,

$$(X + 2Y)(X - 2Y) = 448.$$

Therefore, by Lemma 2.5 and Remark 2.7, we have the following:

$$X = \frac{N_1 + N_2}{2}, \qquad Y = \frac{N_1 - N_2}{4}, \qquad k = \frac{X - 22}{4}, \qquad n = \frac{Y + 2k - 3}{14},$$

where (N_1, N_2) are the factor pairs of $N = 448$. Using all these information, we can now solve for the integer values of n and k, by considering all the factor pairs of 448, as shown in Table 1.

N_1	N_2	X	Y	n	k
1	448	224.5	-111.75	-0.964285714	50.625
2	224	113	-55.5	-0.928571429	22.75
4	112	58	-27	-0.857142857	9
7	64	35.5	-14.25	-0.75	3.375
8	56	32	-12	-0.714285714	2.5
14	32	23	-4.5	-0.5	0.25
16	28	22	-3	-0.428571429	0
448	1	224.5	111.75	15	50.625
224	2	113	55.5	7	22.75
112	4	58	27	3	9
64	7	35.5	14.25	1.285714286	3.375
56	8	32	12	1	2.5
32	14	23	4.5	0.142857143	0.25
28	16	22	3	0	0
-1	-448	-224.5	111.75	-1.035714286	-61.625
-2	-224	-113	55.5	-1.071428571	-33.75
-4	-112	-58	27	-1.142857143	-20
-7	-64	-35.5	14.25	-1.25	-14.375
-8	-56	-32	12	-1.285714286	-13.5
-14	-32	-23	4.5	-1.5	-11.25
-16	-28	-22	3	-1.571428571	-11
-448	-1	-224.5	-111.75	-17	-61.625
-224	-2	-113	-55.5	-9	-33.75
-112	-4	-58	-27	-5	-20
-64	-7	-35.5	-14.25	-3.285714286	-14.375
-56	-8	-32	-12	-3	-13.5
-32	-14	-23	-4.5	-2.142857143	-11.25
-28	-16	-22	-3	-2	-11

Table 1: Solutions for $7n^2 + 3n - 2nk - 2k = 0$.

Table 1 shows that there are 4 possible solutions for the Diophantine equation $7n^2 + 3n - 2nk - 2k = 0$. These are: (n,k)= (3,9), (-5,-20), (0,0), and (-2,-11). However, since we are only concern with possible values of n for wheel graph W_n, then we can limit the value of n to be positive. Hence, from our 4 solutions, only (3,9), where $n = 3$ will satisfy this condition. Therefore, we have shown that W_3 is the only wheel graph that might be edge-graceful by the contrapositive of Lo's Theorem.

To complete the proof, an actual edge-graceful labeling of W_3 is provided in Figure 4.

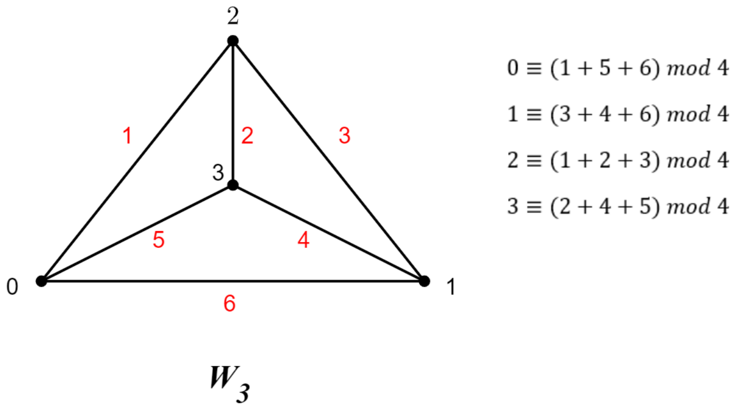

Figure 4: An Edge-graceful labeling of W_3

The notes at the right side of the labeled-graph in Figure 4 shows that it satisfies the definition of an edge-graceful labeling. Since all needed conditions are satisfied, then we can now conclude that W_3 is the only edge-graceful wheel graph. □

4. Conclusion and Future Works

In this paper, the edge-gracefulness of wheel graphs were investigated and determined using their respective vertex sets and edge sets, Lo's Theorem, and the concept of divisibility and Diophantine equation. The investigation lead to a complete proof that W_3 is the only edge-graceful wheel graph, a result that first appeared in the published work of S.Venkatesan and P.Sekar in 2017. A possible continuation to this paper is the determination of edge-graceful graphs in some families of graphs.

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